[Full Names]

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[Course Unit]

[Date]

Coursework problems

Solution 1

A polynomial is monic if the coefficient of its highest degree term is 1(Bihun, Oksana, and Calogero, 1012). Chebyshev polynomials are defined as follows:

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$.

Let n = 2. Then

$T_3(x) = 2xT_2(x) - T_1(x)$

$2x(2x^2 - 1) - x$

$4x^3 - 3x$

Let n = 3. Then

$T_4(x) = 2xT_3(x) - T_2(x)$

$2x(4x^3 - 3x) - (2x^2 - 1)$

$8x^4 - 8x^2 + 1$

For n = 2, the coefficient of the highest degree term x^3 is 4 while for n = 3, the coefficient of x^4 is 8. In both cases, the coefficient is greater than 1, proving that they are not monic polynomials.

Based on the minimum size property, $|_{Tn}(x)| \leq 1$, $-1 \leq x \leq 1$ for all $n \geq 0$.

For $n \geq 1$, $T_n(x) = 2^{n-1}x^n$ + lower degree terms

Solution 2

$Q_n(x) = 2^{1-n}T_n(x) = 2^{1-n}\cos(n\cos^{-1}(x))$ and

$|\text{Cos}(\theta)| \leq \cos(0) = 1$ for all θ

Let $x_j = cos\left(\frac{2j+1}{2n+2}\pi\right)$, j = 0,1,…………, n.

Then $(x - x_1)\cdots(x - x_n) = Q_n(x) \equiv 2^{1-n}T_n(x)$

Proof:

The proof is based on the Fundamental Theorem of Algebra. Each of the x_j is a distinct root of $Q_n(x)$ by construction. The monic polynomial is of degree n. Based on the Fundamental Theorem of Algebra, $Q_n(x)$ must therefore factorize as $Q_n(x) = (x - x_1)\cdots(x - x_n)$

Solution 3

If P(x) is a monic polynomial of degree n, then max $-1 \le x \le 1|P(x)| \ge 2^{1-n}$

Proof.

Suppose that $|P(x)| < 2^{1-n}, \forall x \in \vee -1, 1 \vee$

Set $Q_n(x) = 2^{1-n}T_n(x)$

$x_i = cos\left(\frac{i\pi}{n}\right) fori = 0, 1, \ldots\ldots..n$

By Construction, $Q_n(x)$ is a monic polynomial of degree n, and we will have

$(-1)^i Q_n(x_i) = (-1)^i 2^{1-n} T_n\left(cos\left(\frac{i\pi}{n}\right)\right) = 2^{1-n}(-1)^i(-1)^i = 2^{1-n}$

Since P(x) and $Q_n(x)$ both have a leading coefficient of 1, their difference $Q_n(x)$ - P(x) will be a polynomial of degree less than n – 1. On the other hand,

$(-1)^i P(x_i) \le \left|f(x_i)\right| < 2^{1-n} T_n\left(cos\left(\frac{i\pi}{n}\right)\right) = (-1)^i Q_n(x_i), i = 0, 1, 2, \ldots\ldots\ldots.., n$

Hence,

$(-1)^i|Q_n(x_i) - P(x_i)| > 0$, i=0,1,2,…………,n

Solution 4

If f(x) is a monic polynomial of degree n, then max $-1 \le x \le 1|P(x)| \ge 2^{1-n}$

Proof.

Suppose that $|f(x)| < 2^{1-n}, \forall x \in \vee -1, 1 \vee$

Set $Q_n(x) = 2^{1-n}T_n(x)$

$x_i = cos\left(\frac{i\pi}{n}\right) for i = 0, 1, \ldots\ldots. n$

By Construction, $Q_n(x)$ is a monic polynomial of degree n, and we will have

$(-1)^i Q_n(x_i) = (-1)^i 2^{1-n} T_n\left(cos\left(\frac{i\pi}{n}\right)\right) = 2^{1-n}(-1)^i(-1)^i = 2^{1-n}$

Since f(x) and $Q_n(x)$ both have a leading coefficient of 1, their difference

$Q_n(x)$ - f(x) will be a polynomial of degree less than n – 1. On the other hand,

$(-1)^i P(x_i) \leq |f(x_i)| < 2^{1-n} T_n\left(cos\left(\frac{i\pi}{n}\right)\right) = (-1)^i Q_n(x_i), i = 0, 1, 2, \ldots\ldots\ldots.., n$

Hence,

$(-1)^i|Q_n(x_i) - f(x_i)| > 0$, i=0,1,2,…………,n

Thus, the difference between the two functions $Q_n(x)$ - f(x) ought to oscillate at least n + 1 times over the interval [-1, 1], but this is not possible because the difference is a polynomial of the degree of at most n -1 (Ghosh, Mrinalkanti, & Nimavat). Therefore, there is a contradiction if $|f(x)| < 2^{1-n}, \forall x \in \vee -1, 1 \vee$. The opposite inequality must hold.

If $Q_n(x)$ is monic and defined by $Q^n(x) = 2^{1-n}T^n(x)$, the maximum value of |Qn(x)| on the interval [-1,1] is 2^{-1-n}.

Solution 5

The expression in (2) can be bounded by 2^{-n}.

If the nodes x_j are chosen as the roots of the Chebyshev polynomial $T_{n+1}(x)$

$x_j = cos\left(\frac{2j+1}{2n+2}\pi\right)$, j = 0, 1,…………, n.,

then the error term for the polynomial using the nodes x_j is

$E(x) = f(x) - P(x) \leq \frac{1}{2^n(n+1)!} max -1 \leq t \leq 1 \vee f^{(n+1)}(t)$

This is the best upper bound that is achievable by varying $x_{j.}$

Works Cited

Bihun, Oksana, and Francesco Calogero. "Generations of monic polynomials such that the coefficients of each polynomial of the next generation coincide with the zeros of a polynomial of the current generation, and new solvable many-body problems." *Letters in Mathematical Physics* 106.7 (2016): 1011-1031.

Ghosh, Mrinalkanti, and Rachit Nimavat. "Upper and Lower Bounds on ε-approximate Degree of ANDn and ORn Using Chebyshev Polynomials." (2016).